PROBABILITY

Classical Definition of Probability

$$P(A) = \frac{number\ of\ outcomes\ in\ A}{total\ number\ of\ outcomes}$$

Probability is the measure of how likely an event is.

An **experiment** is a situation involving chance or probability that leads to results called outcomes.

An **outcome** is the result of a single trial of an experiment.

An **event** is one or more outcomes of an experiment.

Though you probably have not seen this definition before, you probably have an inherent grasp of the concept. In other words, you could guess the probabilities without knowing the definition.

Cards and Dice The examples that follow require some knowledge of cards and dice. Here are the basic facts needed compute probabilities concerning cards and dice.

A standard deck of cards has four suites: hearts, clubs, spades & diamonds. Each suite has thirteen cards: ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen and king. Thus, the entire deck has 52 cards total.

When you are asked about the probability of choosing a certain card from a deck of cards, you assume that the cards have been well-shuffled, and that each card in the deck is visible, though face down, so you do not know what the suite or value of the card is.

A pair of *dice* consists of two cubes with dots on each side. One of the cubes is called a *die*, and each die has six sides. Each side of a die has a number of dots (1, 2, 3, 4, 5 or 6), and each number of dots appears only once.

Example 1 The probability of choosing a heart from a deck of cards is given by

$$P(heart) = \frac{number\ of\ hearts}{total\ number\ of\ cards} = \frac{13}{52} = \frac{1}{4} = 0.25$$

Example 2 The probability of choosing a three from a deck of cards is

$$P(three) = \frac{number\ of\ threes}{total\ number\ of\ cards} = \frac{4}{52} = \frac{1}{13} \approx 0.077$$

Example 3 The probability of a two coming up after rolling a die (singular for dice) is

$$P(two) = \frac{number\ of\ twos}{total\ number\ of\ sides} = \frac{1}{6} \approx 0.167$$

Definition of the Fundamental Counting Principle

The *Fundamental Counting Principle* is used to find the number of possible outcomes. It states that if an event has ***m*** possible outcomes and another independent event has ***n*** possible outcomes, then there are ***m•n*** possible outcomes for the two events together.

(If there are m ways to do one thing, and n ways to do another, then there are m•n ways of doing both. The Fundamental Counting Principle is the guiding rule for finding the number of ways to accomplish two tasks.)

Therefore,the number of possible sundaes can also be found using the fundamental counting principle. That is, you multiply the number of choices for each item.

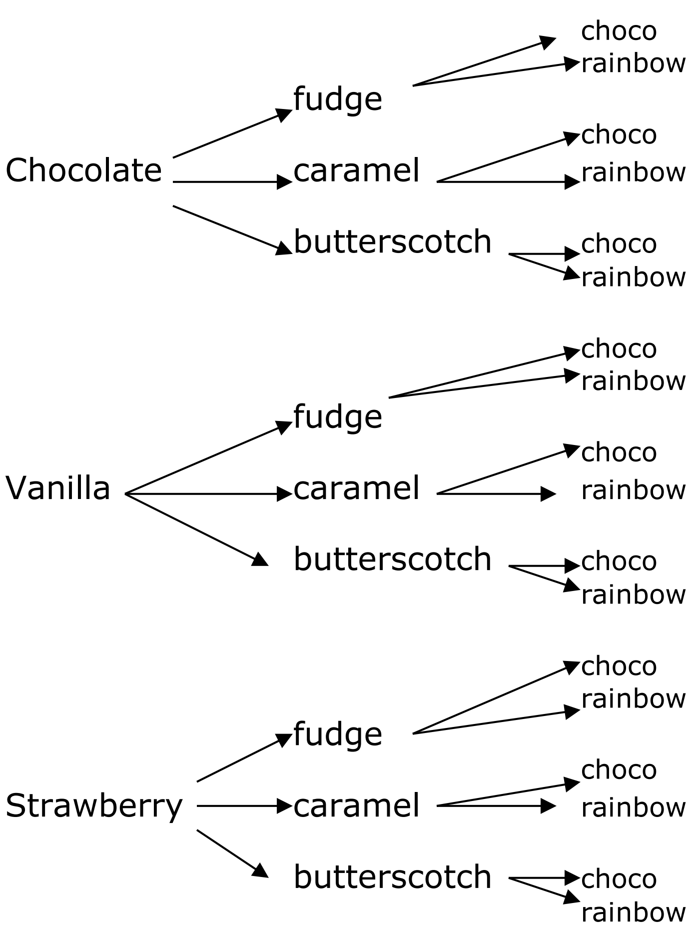
3 flavors • 3 syrups • 2 sprinkles = 18 possible sundaes

<u>Tree Diagram</u>

A listing of all the possible outcomes is called the sample space and is denoted by the capital letter S. A ***tree diagram*** is a graphic organizer used to list all possibilities of a sample space in a systematic way.

Tree diagrams are one method for calculating the total number of outcomes in a sample space. (The sample space shows all the possible outcomes calculated from the fundamental counting principle.)

For example, a tree diagram can illustrate the different single-dip ice cream sundaes that can be made from the ice cream flavors: chocolate, vanilla, and strawberry and the topping choices: fudge, caramel or butterscotch, and then choosing, chocolate or rainbow sprinkles.



The tree diagram shows that there are 18 possible sundaes.

Counting Arrangements

The number of ways that "n things" can be arranged is n!.

The expression n!, read "*n factorial*", where n is greater than zero, is the product of all positive integers beginning with n and counting backward to 1.

For instance, let's say you have 3 pieces of candy: a Reese's Peanut Butter Cup, a Twix and a Milky Way. In how many different ways could you eat them? Diagram the possibilities.

RTM	**RMT**	
TRM	**TMR**	
MRT	**MTR**	…therefore, 6 different ways.

This solution can also be found using $3! = 3 \bullet 2 \bullet 1 = 6$. Factorials are much easier than diagramming when using larger numbers.

Austin and Aly are going to an amusement park. They cannot decide in which order to ride all 12 roller coasters in the park.

~How many different orders can they ride all of the roller coasters if they ride each once?

$$12! = 12 \bullet 11 \bullet 10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 479{,}001{,}600$$

~After standing in line awhile, they realize that they may only be able to ride 8 of the 12 roller coasters. How many ways can they do this?

$$12 \bullet 11 \bullet 10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5 = 19{,}958{,}400$$

Probability of Compound Events

Independent and Dependent Events Compound events are made up of two or more simple events. The events can be **independent events** or they can be **dependent events**.

Probability of Independent Events	Outcome of first event does not affect outcome of second.	$P(A \text{ and } B) =$ $P(A) \cdot P(B)$	Example: rolling a 6 on a die and then rolling a 5
Probability of Dependent Events	Outcome of first event does affect outcome of second.	$P(A \text{ and } B) =$ $P(A) \cdot P(B \text{ following } A)$	Example: without replacing the first card, choosing an ace and then a king from a deck of cards

Example 1: **Find the probability that you will roll a six and then a five when you roll a die twice.**

By the definition of independent events,

$P(A \text{ and B}) = P(A) \cdot P(B)$

First roll: $P(6) = \frac{1}{6}$

Second roll: $P(5) = \frac{1}{6}$

$$P(6 \text{ and } 5) = P(6) \cdot P(5) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

The probability that you will roll a six and then roll a five is $\frac{1}{36}$.

Example 2: **A bag contains 3 red marbles, 2 green marbles, and 4 blue marbles. Two marbles are drawn randomly from the bag and not replaced. Find the probability that both marbles are blue.**

By the definition of dependent events,

$P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$

First marble: $P(\text{blue}) = \frac{4}{9}$

Second marble: $P(\text{blue}) = \frac{3}{8}$

$$P(\text{blue, blue}) = \frac{4}{9} \cdot \frac{3}{8} = \frac{12}{72} = \frac{1}{6}$$

The probability of drawing two blue marbles is $\frac{1}{6}$.

Practice A

A bag contains 3 red, 4 blue, and 6 yellow marbles. One marble is selected at a time, and once a marble is selected, it is not replaced. Find each probability.

1. P(2 yellow) **2.** P(red, yellow) **3.** P(blue, red, yellow)

4. George has two red socks and two white socks in a drawer. What is the probability of picking a red sock and a white sock in that order if the first sock is not replaced?

5. Phyllis drops a penny in a pond, and then she drops a nickel in the pond. What is the probability that both coins land with tails showing?

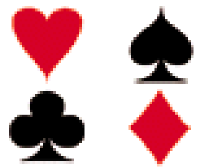
6. A die is rolled and a penny is dropped. Find the probability of rolling a two and showing a tail.

Probability of Compound Events

Mutually Exclusive & Inclusive Events

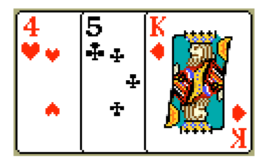
Definition: Two events are **mutually exclusive** if they <u>cannot</u> occur at the same time.

Two events are **mutually inclusive** if one can happen at the same time that another one occurs.



In

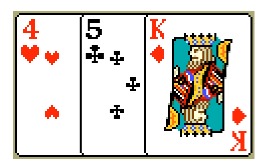
<u>Experiment 1</u>: A single card is chosen at random from a standard deck of 52 playing cards. What is the probability of choosing a 5 or a king?



<u>Possibilities</u>:

1. The card chosen can be a 5.
2. The card chosen can be a king.

<u>Experiment 2</u>: A single card is chosen at random from a standard deck of 52 playing cards. What is the probability of choosing a club or a king?



<u>Possibilities</u>:

1. The card chosen can be a club.
2. The card chosen can be a king.
3. The card chosen can be a king and a club (i.e., the king of clubs).

<u>Experiment 3</u>: A single letter is chosen at random from the word **PROBABILITY**. What is the probability of choosing an **O** or a vowel?

<u>Possibilities</u>:

1. The letter chosen can be an **O**
2. The letter chosen can be a vowel.
3. The letter chosen can be an **O** and a vowel.

In Experiment 1, the card chosen can be a five or a king, *but not both at the same time*. These events are **mutually exclusive**.

In Experiment 2, the card chosen can be a club, or a king, or both at the same time. These events are **not mutually exclusive**, but instead are **mutually inclusive**.

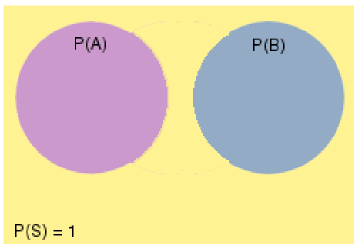
In Experiment 3, these events are **not mutually exclusive** since they *can occur at the same time*. So, they are **mutually inclusive**.

DO YOU REMEMBER VENN DIAGRAMS FROM 6TH GRADE?

Summary: In this lesson, we have learned the difference between mutually exclusive and mutually inclusive events. We can use set theory and Venn Diagrams to illustrate this difference.

Mutually Exclusive Events

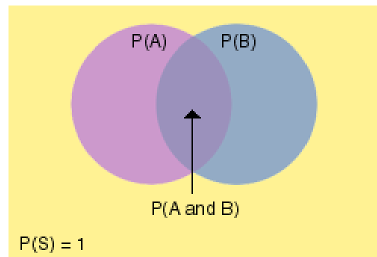
Two events are mutually exclusive if they cannot occur at the same time (i.e., they have no outcomes in common).



In the Venn Diagram above, the probabilities of events A and B are represented by two disjoint sets (i.e., they have no elements in common).

Non-Mutually Inclusive Events

Two events are non-mutually exclusive if they have one or more outcomes in common.



In the Venn Diagram above, the probabilities of events A and B are represented by two intersecting sets (i.e., they have some elements in common).

Note: In each Venn diagram above, the sample space of the experiment is represented by S, with P(S) = 1.

Probability of Mutually Exclusive Events	$P(A \text{ or } B) = P(A) + P(B)$	P(rolling a 2 or a 3 on a die) = $P(2) + P(3) = \frac{1}{3}$
Probability of Inclusive Events	$P(A \text{ or } B) =$ $P(A) + P(B) - P(A \text{ and } B)$	P(king or heart) = $P(\text{K}) + P(\text{H}) - P(\text{K and H}) = \frac{4}{13}$

Example 1: **A card is drawn from a standard deck of playing cards.**

a. Find the probability of drawing a king or a queen.

Drawing a king or a queen are mutually exclusive events. Use the formula for the probability of mutually exclusive events, $P(A \text{ or } B) = P(A) + P(B)$.

$P(A) = P(\text{king}) = \frac{4}{52}$ or $\frac{1}{13}$

$P(B) = P(\text{queen}) = \frac{4}{52}$ or $\frac{1}{13}$

$P(\text{king or queen}) = \frac{1}{13} + \frac{1}{13}$ or $\frac{2}{13}$

The probability of drawing a king or a queen is $\frac{2}{13}$.

Example 2: **Find the probability of drawing a face card or a spade.**

Since it is possible to draw a card that is a face card and a spade, these are inclusive events. Use the formula

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

$P(A) = P(\text{face card}) = \frac{12}{52}$

$P(B) = P(\text{spade}) = \frac{13}{52}$

$P(A \text{ and } B) = P(\text{face card and spade}) = \frac{3}{52}$

$P(\text{face card or spade}) = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52}$ or $\frac{11}{26}$

The probability of drawing a face card or a spade is $\frac{11}{26}$.

Practice B

A bag contains 2 red, 5 blue, and 7 yellow marbles. Find each probability.

1. P(yellow or red) **2.** P(red or not yellow) **3.** P(blue or red or yellow)

A card is drawn from a standard deck of playing cards. Find each probability.

4. P(jack or red) **5.** P(red or black) **6.** P(jack or clubs)

7. P(queen or less than 3) **8.** P(5 or 6) **9.** P(diamond or spade)

10. In a math class, 12 out of 15 girls are 14 years old and 14 out of 17 boys are 14 years old. What is the probability of selecting a girl or a 14-year old from this class?

SOLUTIONS:

Practice A

1. **5/26** 2. **3/26** 3. **6/143**
4. **1/3** 5. **1/4** 6. **1/12**

Practice B

1. **9 /14** 2. **1/2** 3. 1
4. **7/ 13** 5. 1 6. **4/ 13**
7. **3/ 1**3 8. **2/ 13** 9. **1/2**
10. **29/ 32**

Combinations and Permutations

What's the Difference???

In English we use the word "*combination*" loosely,
without thinking if the **order** of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas"
We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe was 472". Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

⇒ If the order doesn't matter, it is a **Combination**.

⇒ If the order **does** matter it is a **Permutation**.

So, we should really call this a "Permutation Lock"!

In other words:

A Permutation is an **ordered** Combination.

To help you to remember, think "**P**ermutation ... **P**osition"

Permutations

There are basically two types of permutation:

1. **Repetition is Allowed**: such as the lock above. For instance, it could be "333".
2. **No Repetition**: for example the first three people in a running race;
you can't be first *and* second.

1. Permutations <u>with Repetition</u>

These are the easiest to calculate. When you have ***n*** things to choose from you have ***n*** choices each time! When choosing ***r*** of them, the permutations are:

n × n × ... *(r times)*

(In other words, there are **n** possibilities for the first choice,
THEN there are **n** possibilites for the second choice, and so on, multplying each time.)

This is easiest to write down using an exponent of **r**:

n × n × ... (r times) = n^r

For example: in the lock above, there are 10 numbers to choose from (0 through 9) and you choose 3 of them: **10 × 10 × ... (3 times) = 10^3 = 1,000 permutations**
So, the formula is simply: **n^r ,** where ***n*** is the number of things to choose from, and you choose ***r*** of them (Repetition allowed, **BUT** order matters).

2. Permutations <u>without Repetition</u>

In this case, you have to **reduce** the number of available choices each time.

For example, what order could 16 pool balls be in? After choosing, say, number "14" you can't choose it again.

So, your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, then 13, and so on. And the total permutations would be:

16 × 15 × 14 × 13 × ... = 20,922,789,888,000

But how do we write that mathematically? Answer: we use the "factorial notation"

The **factorial notation** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

- 4! = 4 × 3 × 2 × 1 = 24
- 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5,040
- 1! = 1

So, if you wanted to select **all** of the billiard balls, the permutations would be:

16! = 20,922,789,888,000

What if you wanted to select just 3, then you have to stop the multiplying after 14.

How do you do that? There is a neat trick ... you divide by **13!** ...

$$\frac{16 \times 15 \times 14 \times 13 \times 12 \ldots}{13 \times 12 \ldots} = 16 \times 15 \times 14 = 3{,}360$$

Do you see? $\frac{16!}{13!} = 16 \times 15 \times 14$

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from, and you choose ***r*** of them (*No repetition, order matters*)

Examples:

Our "order of 3 out of 16 billiard balls example" would be:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20{,}922{,}789{,}888{,}000}{6{,}227{,}020{,}800} = 3{,}360$$

(which is just the same as: **16 × 15 × 14 = 3,360**)

{There are 3,360 ways to choose 3 billiard balls out of 16.}

How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3{,}628{,}800}{40{,}320} = 90$$

(which is just the same as: **10 × 9 = 90**)

{There are 90 different ways that 2 out of 10 people can come in first or second place.}

Notation

$$P(n, r) = {}_nP_r = \frac{n!}{(n-r)!}$$

Example: $P(10, 2) = \frac{10!}{8!} = 90$

Combinations

There are also two types of combinations (remember the order does **not** matter now):

1. **Repetition is Allowed**: such as coins in your pocket (5¢,5¢,5¢,10¢,10¢)
2. **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

1. Combinations <u>with Repetition</u>

Actually, these are the hardest to explain and are not 'tackled' in 9th grade mathematics.

2. Combinations <u>without Repetition</u>

This is how lotteries work. The numbers are drawn one at a time, and if you have the lucky numbers (no matter what order) you win!

The easiest way to explain it is to:

- assume that the order does matter (i.e., permutations),
- then alter it so the order does **not** matter.

Going back to our billiards example, let us say you are shooting pool and after you break, 3 balls go into the pockets. You just want to know which 3 pool balls were chosen, not the order. We already know that 3 out of 16 gave us 3,360 permutations, but many of those permutations will be the same to us now, because we don't care what order!

So, let us say the balls numbered 10, 12 and 13 went into the pockets. These are the possibilities:

<u>Permutation</u> Order does matter	<u>Combination</u> Order doesn't matter
10 12 13	
10 13 12	
12 10 13	
12 13 10	10 12 13
13 10 12	
13 12 10	

So, the permutations will have 6 times as many possibilities as the combinations. In fact, there is an easy way to work out how many ways "10, 12, 13" could be placed in order, and we have already talked about it. The answer is: **3! = 3 × 2 × 1 = 6, factorial notation.**

Therefore, all we need to do is adjust our permutations formula and **reduce it** by how many ways the objects could be in order (because we aren't interested in the order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

where ***n*** is the number of things to choose from, and you choose ***r*** of them
(*No repetition, order <u>doesn't</u> matter*)

It is often referred to as, "*n choose r*" (such as "16 choose 3").

Notation

$$C(n, r) = {}_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Example

So now our billiards example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!\times 13!} = \frac{20{,}922{,}789{,}888{,}000}{6\times 6{,}227{,}020{,}800} = 560$$

Or you could do it this way:

$$\frac{16\times 15\times 14}{3\times 2\times 1} = \frac{3360}{6} = 560$$

So remember, do the permutation, then reduce by a further "**r!**" Or better still...
JUST REMEMBER THE FORMULA!

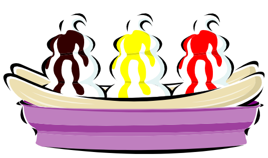
It is interesting to note how this formula is nice and **symmetrical**:

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{13!(16-13)!} = \frac{16!}{3!\times 13!} = 560$$

*1. Combinations with Repetition

OK, now we can tackle this one ...



Let us say there are five flavors of ice cream: **bubble gum, chocolate, oreo strawberry and vanilla**. Three scoops are used to make a sundae.
How many variations will there be?

Let's use letters for the flavors: {b, c, o, s, v}.
Some example selections would be:

- {c, c, c} (3 scoops of chocolate)
- {b, o, v} (one each of bubble gum, oreo and vanilla)
- {v, o, v} (one of oreo, two of vanilla)

(And just to be clear: There are **n=5** things to choose from and you choose **r=3** of them.
Order does not matter, and you **can** repeat!)

Think of it this way, no matter what 3 scoops you order, it really doesn't matter how they are arranged on the sundae dish. You will still get what you want!

$$\frac{(n + r - 1)!}{r!\,(n - 1)!}$$

Where ***n*** is the number of things to choose from and you choose ***r*** of them (Repetition allowed, order doesn't matter)

So, what is the answer to our example?

$$\frac{(5 + 3 - 1)!}{3!(5 - 1)!} = \frac{7!}{3! \bullet 4!} = \frac{5040}{6 \bullet 24} = 35$$

So we see that there will be 35 variations of 3 scoops of ice cream when we have 5 flavors to choose from. Again, this is not a topic or formula you need to know (now), but I'm just sayin'!

In Conclusion

Phew, that was a lot to absorb, so maybe you could read it again to be sure!

Knowing *how* these formulas work is only half the battle.
Figuring out how to interpret a real world situation can be challenging.

At least now you know how to calculate all 4 variations of "Order does/does not matter" and "Repeats are/are not allowed".

Combinations and Permutations
PRACTICE

1. If the NCAA has applications from 6 universities for hosting its intercollegiate tennis championships in 2006 and 2007, how many ways may they select the hosts for these championships

   a) if they are not both to be held at the same university?

   b) if they may both be held at the same university?

2. There are five finalists in the Mr. Rock Hill pageant. In how many ways may the judges choose a winner and a first runner-up?

3. In a primary election, there are four candidates for mayor, five candidates for city treasurer, and two candidates for county attorney. In how many ways may voters mark their ballots

   a) if they vote in all three of the races?

   b) if they exercise their right not to vote in any or all of the races?

4. A multiple-choice test consists of 15 questions, each permitting a choice of 5 alternatives. In how many ways may a student fill in the answers if they answer each question?

5. A television director is scheduling a certain sponsor's commercials for an upcoming broadcast. There are six slots available for commercials. In how many ways may the director schedule the commercials

   a) If the sponsor has six different commercials, each to be shown once?

   b) If the sponsor has three different commercials, each to be shown twice?

   c) If the sponsor has two different commercials, each to be shown three times?

   d) If the sponsor has three different commercials, the first of which is to be shown three times, the second two times, and the third once?

6. In how many ways may can five persons line up to get on a bus?

7. In how many ways may these same people line up if two of the people refuse to stand next to each other?

8. In how many ways may 8 people form a circle for a folk dance?

9. How many permutations are there of the letters in the word "great"?

10. How many permutations are there of the letters in the word "greet"?

11. How many distinct permutations are there of the word "statistics"?

12. How many distinct permutations of the word "statistics" begin and end with the letter "s"?

13. A college football team plays 10 games during the season. In how many ways can it end the season with 5 wins, 4 losses, and 1 tie?

14. If eight people eat dinner together, in how many different ways may 3 order chicken, 4 order steak, and 1 order lobster?

15. Suppose a True-False test has 20 questions.

    a) In how many ways may a student mark the test, if each question is answered?
    b) In how many ways may a student mark the test, if 7 questions are marked correctly and 13 incorrectly?
    c) In how many ways may a student mark the test, if 10 questions are marked correctly and 10 incorrectly?

16. Among the seven nominees for two vacancies on the city council are three men and four women. In how many ways may these vacancies be filled

    a) with any two of the nominees?
    b) with any two of the women?
    c) with one of the men and one of the women?

17. Mr. Jones owns 4 pairs of pants, 7 shirts, and 3 sweaters. In how many ways may he choose 2 of the pairs of pants, 3 of the shirts, and 1 of the sweaters to pack for a trip?

18. In how many ways may one A, three B's, two C's, and one F be distributed among seven students in a CTQR 150 class?

19. An art collector, who owns 10 original paintings, is preparing a will. In how many ways may the collector leave these paintings to three heirs?

20. A baseball fan has a pair of tickets to 6 different home games of the Chicago Cubs. If the fan has five friends who like baseball, how many ways may he take one of them along to each of the six games?

Combinations and Permutations
PRACTICE-SOLUTIONS

1. a) $6 \cdot 5 = 30$ b) $6 \cdot 6 = 36$
2. $5 \cdot 4 = 20$
3. a) $4 \cdot 5 \cdot 2 = 40$ b) $5 \cdot 6 \cdot 3 = 90$
4. $5^{15} = 30,517,578,125$
5. a) $6! = 720$ b) $C(6,2) \cdot C(4,2) = 90$ c) $C(6,3) = 20$ d) $C(6,3) \cdot C(3,2) = 60$
6. $P(5,5) = 5! = 120$
7. $5! - 4 \cdot 2 \cdot 3! = 72$
8. $\dfrac{8!}{8} = 7! = 5040$
9. $5! = 120$
10. $\dfrac{5!}{2!} = 60$
11. $\dfrac{10!}{3!3!2!} = 50,400$
12. $\dfrac{8!}{3!2!} = 3360$
13. $C(10,5) \cdot C(5,4) = 1260$
14. $C(8,3) \cdot C(5,4) = 280$
15. a) $2^{20} = 1,048,576$ b) $C(20,7) = 77,520$ c) $C(20,10) = 184,756$
16. a) $C(7,2) = 21$ b) $C(4,2) = 6$ c) $C(4,1) \cdot C(3,1) = 12$
17. $C(4,2) \cdot C(7,3) \cdot C(3,1) = 630$
18. $C(7,1) \cdot C(6,3) \cdot C(3,2) = 420$
19. $3^{10} = 59,049$
20. $5^6 = 15,625$